VARIANT OF GENERALIZATION OF A. A. II'YUSHIN'S THEORY OF ELASTOPLASTIC PROCESSES FOR THE CASE OF LARGE PLASTIC STRAINS

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One of the trends in the theory of plasticity in recent decades has been an increasing interest in geometrically nonlinear problems. The central problem in the geometrically nonlinear theory of plasticity (NTP) remains establishment of the governing relations. In most investigations (see [1-5], for example) devoted to the construction of governing relations of the NTP, this goal is approached through the use of equations of the theory of plastic flow based on the gradient principle and some strain-hardening law (kinematic, isotropic, or a combination of the two). To satisfy the requirement of objectivity in the flow theory equations, substantial derivatives of measures of the stress and strain states are replaced by certain objective derivatives of these measures. However, as is known [6, 7, etc.], the above-noted relations reflect deformations along paths of small curvature with an acceptable degree of accuracy. Here we examine one possible variant of generalization of the theory of elastoplastic strains developed by A. A. Il'yushin and associated special theories of plasticity suitable for describing deformation over complex paths. All of the theories are examined for the case of large plastic strains.

1. Certain Kinematic Relations. We will mainly use tensorial (symbolic) notation for the relations. The position vectors of a material particle with the Lagrangian coordinates (ξ^1, ξ^2, ξ^3) in the reference (at the moment of time t_0) configuration $\mathscr{K}_0$ and the actual configuration $\mathscr{K}_t$ are designated respectively as $\mathbf{R}_0(\xi^i)$ and $\mathbf{r}(\xi^i, t)$, while the gradients of the position in $\mathscr{K}_0$ and $\mathscr{K}_t$ are designated as $\widehat{\nabla}\mathbf{R}_0$ and $\mathring{\nabla}\mathbf{r}$.

We will henceforth employ polar expansion of the position gradient [8, 9]

$$\mathring{\nabla}\mathbf{r} = \mathbf{U}\cdot\mathbf{R} = \mathbf{R}\cdot\mathbf{V}, \tag{1.1}$$

where $\mathbf{U} = \sum_{i=1}^{3} U_i \mathring{\mathbf{p}}^i \mathring{\mathbf{p}}_i$, $\mathbf{V} = \sum_{i=1}^{3} V_i \widehat{\mathbf{p}}^i \widehat{\mathbf{p}}_i$ are the left and right tensors of the strains, respectively; $\mathring{\mathbf{p}}_i = \mathring{\mathbf{p}}^i$ and $\widehat{\mathbf{p}}_i = \widehat{\mathbf{p}}^i$ are their principal vectors;

$$\mathbf{R} = \mathring{\mathbf{p}}_i \widehat{\mathbf{p}}^i, \ \mathbf{R}^T = \widehat{\mathbf{p}}_i \mathring{\mathbf{p}}^i \tag{1.2}$$

are the rotation tensor and its transport. As the strain measures, we take the Hencky strain tensors $\mathring{\mathbf{H}} = \sum_{i=1}^{3} \ln U_i \mathring{\mathbf{p}}^i \mathring{\mathbf{p}}_i$, $\widehat{\mathbf{H}} = \sum_{i=1}^{3} \ln V_i \widehat{\mathbf{p}}^i \widehat{\mathbf{p}}_i$, as the measure of the stress state, we take the Cauchy stress tensor σ [8, 9].

Along with measures of the stress and strain states, the governing relations of the NTP also make use of measures of the rates of their change. For example, the substantial derivatives $\dot{\sigma}$ and $\dot{\widehat{\mathbf{H}}}$ cannot be used as rate measures because they cannot be further differentiated [9, 10]. The selection of objective (indifferent) measures of the rates of change in the stresses and strains requires analysis of their geometric sense and is connected with the method used to expand the motion into quasirigid ("rigid") and deformational motions. Such an expansion can be accomplished by many different methods. As was shown in [10], the types of indifferent derivatives used in the NTP are the relative rates of change of the respective tensors in the chosen moving (and, generally speaking, deformable) coordinate system. Here, it is usually assumed that the motion of the latter describes quasirigid motion. In the

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present study, we employ a polar expansion of the position gradient (1.1) to distinguish quasirigid motion.

We will examine a material particle M with a small neighborhood in which the stress-strain state (SSS) can be considered uniform. For simplicity, we make the Cartesian coordinate system $Ox^1x^2x^3$ with the orthonormal basis $\mathbf{k}_i = \mathbf{k}^i$, $i = \overline{1,3}$. Together with the Lagrangian systems $O\xi^1\xi^2\xi^3$ with the bases $\overset{\circ}{\mathbf{e}}_i$ (in $\mathscr{K}_0$) and $\widehat{\mathbf{e}}_i$ (in $\mathscr{K}_t$), we introduce the moving local coordinate system $M\zeta^1\zeta^2\zeta^3$. The latter system will also be assumed to be an orthogonal Cartesian system, with the basis $\mathbf{q}_i = \mathbf{q}^i$. In the reference configuration $\mathscr{K}_0$ we will assume that the vectors of the basis $\mathbf{q}_i$ coincide with $\mathbf{k}_i$, $\mathbf{q}_i|_{t=t_0} \equiv \mathbf{q}_i^0 = \mathbf{k}_i$, $i = \overline{1, 3}$.

In accordance with the method being employed to expand the motion of a particle with a neighborhood, the quasirigid motion of the particle is described by the motion of the system $M\zeta^1\zeta^2\zeta^3$. The orientation of the latter relative to the system of reference at each moment of time is determined by the rotation tensor R:

$$\mathbf{q}_i = \mathbf{R}^T\cdot\mathbf{q}_i^0 = \mathbf{q}_i^0\cdot\mathbf{R} = \mathbf{k}_i\cdot\mathbf{R}, \quad i = \overline{1, 3}. \tag{1.3}$$

It is not hard to see from Eqs. (1.3) that the rotation tensor can be determined in a manner different from (1.2):

$$\mathbf{R} = \mathbf{q}_i^0\mathbf{q}^i, \quad \mathbf{R}^T = \mathbf{q}_i\mathbf{q}^{0i}. \tag{1.4}$$

In (1.2), each vector entering into the dyad product changes its orientation in the general case of motion, while in (1.4) the vector $\mathbf{q}_i^0$ is fixed in relation to the material fibers in $\mathscr{K}_0$. At each moment of time, the motion of the system $M\zeta^1\zeta^2\zeta^3$ can be represented by an infinitesimal translational displacement at the velocity $\mathbf{v}_M$ and an infinitesimal rotation with an angular velocity corresponding to the spin tensor $\mathbf{\Omega}$ [9]:

$$\mathbf{\Omega} = \dot{\mathbf{R}}^T\cdot\mathbf{R} = -\mathbf{R}^T\cdot\dot{\mathbf{R}} = -\mathbf{\Omega}^T. \tag{1.5}$$

With allowance for the above remarks, it is not hard to show from (1.4) and (1.5) that $\mathbf{\Omega} = \dot{\mathbf{q}}_i\mathbf{q}^i$, so that

$$\dot{\mathbf{q}}_i = \mathbf{\Omega}\cdot\mathbf{q}_i. \tag{1.6}$$

The choice of the moving coordinate system corresponding to quasirigid motion unambiguously determines the deformational motion: by deformational motion, we mean the motion of the medium relative to the moving system that is introduced. Here, the measures of the rates of change in the stress and strain states are the relative rates of change in the measures of these states. Given the method of motion expansion being used an the present investigation, the relative rates of change of σ and $\widehat{\mathbf{H}}$ are the so-called Zaremby derivatives [11]:

$$\sigma^Z = \dot{\sigma} + \sigma\cdot\mathbf{\Omega} - \mathbf{\Omega}\cdot\sigma; \tag{1.7}$$

$$\widehat{\mathbf{H}}^Z = \dot{\widehat{\mathbf{H}}} + \widehat{\mathbf{H}}\cdot\mathbf{\Omega} - \mathbf{\Omega}\cdot\widehat{\mathbf{H}}. \tag{1.8}$$

Assuming that the eigenvectors $\overset{\circ}{\mathbf{p}}_i$, $\widehat{\mathbf{p}}_i$ of the tensors $\overset{\circ}{\mathbf{H}}$, $\widehat{\mathbf{H}}$ are continuous vector-functions of time with continuous first derivatives, we can determine the set of three orthonormalized vectors $\mathbf{p}_i = \lim\limits_{t\to t_0}\overset{\circ}{\mathbf{p}}_i = \lim\limits_{t\to t_0}\widehat{\mathbf{p}}_i$. The vectors $\mathbf{p}_i$ for each deformation process are fixed in relation to the material fibers in $\mathscr{K}_0$ and, as can shown, are the principal vectors of the strain-rate tensor $\mathbf{D}$ at the moment t_0. Then the rotation tensors $\mathbf{R}_U$ and $\mathbf{R}_V$ of the eigenvectors $\mathbf{R}_U$ and $\mathbf{R}_V$ for their movement from the initial to the current position can be determined by the relations

$$\mathbf{R}_U = \mathbf{p}_k\overset{\circ}{\mathbf{p}}{}^k, \quad \mathbf{R}_U^T = \overset{\circ}{\mathbf{p}}_k\mathbf{p}^k, \quad \mathbf{R}_V = \mathbf{p}_k\widehat{\mathbf{p}}{}^k, \quad \mathbf{R}_V^T = \widehat{\mathbf{p}}_k\mathbf{p}^k, \tag{1.9}$$

here

$$\mathbf{R} = \mathbf{R}_U^T\cdot\mathbf{R}_V, \quad \mathbf{R}^T = \mathbf{R}_V^T\cdot\mathbf{R}_U. \tag{10}$$

We express the spins $\mathbf{\Omega}_U$ and $\mathbf{\Omega}_V$ of the triads $\overset{\circ}{\mathbf{p}}_i$ and $\widehat{\mathbf{p}}_i$. in the form

$$\mathbf{\Omega}_U = \dot{\mathbf{R}}_U^T\cdot\mathbf{R}_U, \quad \mathbf{\Omega}_V = \dot{\mathbf{R}}_V^T\cdot\mathbf{R}_V, \tag{1.11}$$

so that

$$\dot{\overset{\circ}{\mathbf{p}}}_i = \mathbf{\Omega}_U\cdot\overset{\circ}{\mathbf{p}}_i = -\overset{\circ}{\mathbf{p}}_i\cdot\mathbf{\Omega}_U, \quad \dot{\widehat{\mathbf{p}}}_i = \mathbf{\Omega}_V\cdot\widehat{\mathbf{p}}_i = -\widehat{\mathbf{p}}_i\cdot\mathbf{\Omega}_V. \tag{1.12}$$

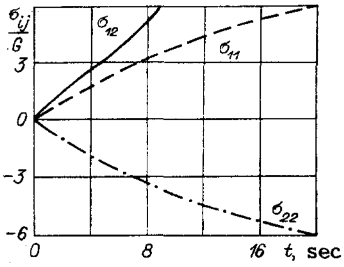

Fig. 1

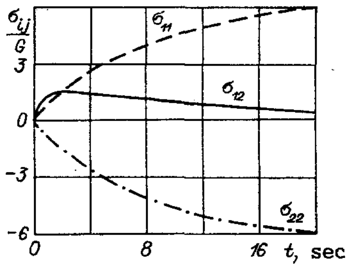

Fig. 2

Using (1.5), (1.10), and (1.11)

$$(1.13)$$

With allowance for (1.12), we use Eq. (1.8) and representation of the tensors $\overset{\circ}{\mathbf{H}}$, $\widehat{\mathbf{H}}$ through eigenvalues and eigenvectors to obtain

$$\dot{\overset{\circ}{\mathbf{H}}} = \sum_{i=3}^{3} \dot{\overset{\circ}{\mathbf{H}}}_i \overset{\circ}{\mathbf{p}}{}^i \overset{\circ}{\mathbf{p}}_i + \Omega_U\cdot\overset{\circ}{\mathbf{H}} - \overset{\circ}{\mathbf{H}}\cdot\Omega_U; \tag{1.14}$$

$$\widehat{\mathbf{H}}^Z = \sum_{i=3}^{3} \dot{\widehat{H}}_i \widehat{\mathbf{p}}{}^i \widehat{\mathbf{p}}_i + \widehat{\mathbf{H}}\cdot(\Omega - \Omega_V) - (\Omega - \Omega_V)\cdot\widehat{\mathbf{H}}. \tag{1.15}$$

By virtue of the equalities $U_i = V_i\ \forall t,\ i = \overline{1,3},$ we have $\overset{\circ}{H}_i = \widehat{H}_i\ \forall t, i = \overline{1,3},$ so that (assuming $\overset{\circ}{H}_i,\ \widehat{H}_i \in C^1([0, \infty))\ \forall i = \overline{1,3}$), $\dot{\overset{\circ}{H}}_i = \dot{\widehat{H}}_i\ \ \forall t, i = \overline{1,3}$. We also write the obvious relations between $\overset{\circ}{H}$ and $\widehat{H}$:

$$\overset{\circ}{\mathbf{H}} = \mathbf{R}\cdot\widehat{\mathbf{H}}\cdot\mathbf{R}^T,\ \widehat{\mathbf{H}} = \mathbf{R}^T\cdot\overset{\circ}{\mathbf{H}}\cdot\mathbf{R}\ \forall t, \tag{1.16}$$

from which it follows in particular that $I_1(\overset{\circ}{\mathbf{H}}) = I_1(\widehat{\mathbf{H}})\ \forall t$, where $I_1(\cdot)$ denotes the first invariant of the corresponding tensor. For practical application, it is convenient to use the expression of $\widehat{\mathbf{H}}^Z$ presented in [10]. Here, this quantity is expressed through the current parameters of the deformation process (velocity gradient, eigenvalues, and eigenvectors U and V). The expression ln [10] is presented below together with the expression for the spin $\mathbf{\Omega}$:

$$\widehat{\mathbf{H}}^Z = \mathbf{D} + \sum_{\substack{i,k=1\\ i\neq k}}^{3} \left\{\left[\frac{2V_kV_i}{V_k^2 - V_i^2}\ln\left(\frac{V_k}{V_i}\right) - 1\right](\widehat{\mathbf{p}}{}^i\cdot\mathbf{D}\cdot\widehat{\mathbf{p}}{}^k)\,\widehat{\mathbf{p}}_k\widehat{\mathbf{p}}_i\right\}; \tag{1.17}$$

$$\Omega = \mathbf{W} + \sum_{i,k=1}^{3}\left[\frac{V_i - V_k}{V_i + V_k}(\widehat{\mathbf{p}}_i\cdot\mathbf{D}\cdot\widehat{\mathbf{p}}_k)\,\widehat{\mathbf{p}}{}^k\widehat{\mathbf{p}}{}^i\right], \tag{1.18}$$

where **W** is the rotation tensor; **D** is the strain-rate tensor.

Note 1. In the strain theory of the mechanics of deformable solids, there are strain measures having the same eigenvalues but different eigenvector triads ($\overset{\circ}{\mathbf{p}}_i$ and $\widehat{\mathbf{p}}_i$), which are transformed using the rotation tensor **R**. Such measures (we will refer to them as associated measures) include the left **U** and right **V** strain tensors, the logarithmic measures $\overset{\circ}{\mathbf{H}}$ and $\widehat{\mathbf{H}}$ the Cauchy–Green measure $\overset{\circ}{\mathbf{G}}$ and Finger measure $\widehat{\mathbf{G}}^{-1} = \mathbf{F}$, and the inverse Cauchy–Green measure $\overset{\circ}{\mathbf{G}}$ and Almansi measure $\widehat{\mathbf{G}}$. Designating the first measure in each pair as $\overset{\circ}{\mathbf{A}}$, and the second as $\widehat{\mathbf{A}}$, it is not hard to show that a relation of the type (1.16) is valid for each pair:

$$\overset{\circ}{\mathbf{A}} = \mathbf{R}\cdot\widehat{\mathbf{A}}\cdot\mathbf{R}^T, \widehat{\mathbf{A}} = \mathbf{R}^T\cdot\overset{\circ}{\mathbf{A}}\cdot\mathbf{R}. \tag{1.16'}$$

Using the rule of tensor transformation with rigid motion, we can state the following: if the tensor $\overset{\circ}{\mathbf{A}}$ is invariant relative to rigid motion [10], then the tensor $\widehat{\mathbf{A}}$ is indifferent, and conversely.

Note 2. The orthogonal tensor **R** accompanying deformation was used by V. I. Levitas (see [12], for example) to construct governing relations, where its application was based on other considerations: it was assumed that the relationship between the Cauchy stress tensor, its material derivative, and the strain-rate tensor was known in the case of deformation "without rotation." Then a rigid displacement was superimposed on this motion. Here, the rigid rotation was assumed to have corresponded to the rotation tensor, which in turn leads to corresponding transformations of the initial equations to a form satisfying the requirement of indifference.

2. Representation of the Loading Process. One of the most important concepts in the theory of elastoplastic processes is that of the representation of the loading process [13]. In the case of small strains (and rotations), the strain path and the stress vector are determined by the components of the corresponding deviators in the basis of the system of reference. However, when indifferent strain and stress measures are used, the image of the loading process constructed in this manner is indifferent (and is noninvariant relative to rigid motion) [14]. This makes similar representation of the loading process unsuitable for the NTP.

One possible method of constructing an image of the process is to use components of the stress and strain measures, invariant to rigid motion, in the basis of the Lagrangian coordinate system of reference. However, the well-known stress tensors [8, 9] defined in terms of $\mathcal{K}_0$ do not have as clear a physical significance as the Cauchy stress tensor $\boldsymbol{\sigma}$. This naturally creates problems in interpreting the necessary experimental data. Also, the first invariants of these stress tensors (such as the second Piola–Kirchhoff tensor) do not characterize the mean stress, which nullifies the physical significance of expansion of stress tensors into spherical and deviatoric parts. It should be noted that the two remaining invariants of the above-noted stress tensors also do not have the significance of the corresponding invariants of the tensor $\boldsymbol{\sigma}$ [13].

In connection with this, here we propose another method of constructing the strain path and stress vector. This method makes use of the Hencky strain tensor $\widehat{\mathrm{H}}$. and the Cauchy stress tensor $\boldsymbol{\sigma}$. The components of these tensors, in terms of the actual configuration $\mathcal{K}_t$ have a distinct physical meaning.

It is customary [13] to introduce vector spaces for the strains B^5 and stresses Σ^5. However, the vectors of the strains b and stresses Σ are found from components of the strain deviator $\widehat{\mathrm{h}}=\widehat{\mathrm{H}}-\frac{1}{3}I_1(\widehat{\mathrm{H}})\mathrm{E}$ and stress deviator $\mathrm{S}=\sigma-\frac{1}{3}I_1(\sigma)\,\mathrm{E}$ in the basis $\mathbf{q}_i$ of the moving coordinate system $M\zeta^1\zeta^2\zeta^3$. We will show that the stress vector is indifferent. To do this, it is sufficient to prove that the components of the tensor $\boldsymbol{\sigma}$ do not change in the basis $\mathbf{q}_i$ with the superposition of an arbitrary rigid motion.

We will examine the motions $\mathbf{r}(\xi^i, t)$ and $\mathbf{r}'(\xi^i, t)$, which differ by a rigid displacement

$$\mathbf{r}'(\xi^i, t)=\mathbf{r}_0'(t)+\left[\mathbf{r}(\xi^i, t)-\mathbf{r}_0(t)\right]\cdot\mathrm{O}(t). \tag{2.1}$$

In this case, the following relations are valid

$$\sigma'(\mathbf{r}')=\mathrm{O}^T\cdot\sigma(\mathbf{r})\cdot\mathrm{O}; \tag{2.2}$$

$$\mathrm{R}'(\mathbf{r}')=\mathrm{R}(\mathbf{r})\cdot\mathrm{O},\ (\mathrm{R}'(\mathbf{r}'))^T=\mathrm{O}^T\cdot(\mathrm{R}(\mathbf{r}))^T. \tag{2.3}$$

We will assume that the reference configuration $\mathcal{K}_0$ is identical for the motions $\mathbf{r}$ and $\mathbf{r}'$. Then

$$\mathbf{q}_i'^0(\mathbf{r}')=\mathbf{q}_i^0(\mathbf{r}),\quad i=\overline{1,3}. \tag{2.4}$$

We recall that the vectors $\mathbf{q}_i^0$ are directed along mutually perpendicular fibers fixed in $\mathcal{K}_0$. Then the components of the tensor $\boldsymbol{\sigma}$ in the bases $\mathbf{q}_i(\mathbf{r})$ and $\mathbf{q}_i'(\mathbf{r}')$ are determined by the relations

$$\begin{gathered}\sigma_{ij}=\mathbf{q}_i\cdot\sigma\cdot\mathbf{q}_j=\mathbf{q}_i^0\cdot\left(\mathrm{R}\cdot\sigma\cdot\mathrm{R}^T\right)\cdot\mathbf{q}_j^0,\\ \sigma_{ij}'=\mathbf{q}_i'\cdot\sigma\cdot\mathbf{q}_j'=\mathbf{q}_i'^0\cdot\left(\mathrm{R}'\cdot\sigma'\cdot\mathrm{R}'^T\right)\cdot\mathbf{q}_i'^0\\ =\mathbf{q}_i^0\cdot\left(\mathrm{R}\cdot\mathrm{O}\cdot\mathrm{O}^T\cdot\sigma\cdot\mathrm{O}\cdot\mathrm{O}^T\cdot\mathrm{R}^T\right)\cdot\mathbf{q}_j^0=\sigma_{ij},\quad i,j=\overline{1,3}.\end{gathered} \tag{2.5}$$

Thus, the stress vector Σ constructed from the components of the tensor $\boldsymbol{\sigma}$ (or the deviator S) in the basis $\mathbf{q}_i$ is indifferent.

We can similarly establish the indifference of the strain path, but we will do this by another method. We first of all note that the strain path constructed from the components of the strain measure is invariant to rigid motion in any stationary basis in and does not change with the superposition of rigid motion. The strain tensor $\widehat{\mathbf{H}}$ associated with $\overset{\circ}{\mathbf{H}}$. can be used to construct the strain path.

THEOREM 1. The strain path constructed using the components of the tensor $\overset{\circ}{\mathbf{H}}$ in the basis $\mathbf{q}_i^0$, coincides with the strain path determined by the components of the tensor $\widehat{\mathbf{H}}$ in the basis $\mathbf{q}_i$. Proof of this theorem reduces to establishment of the relation

$$\mathbf{q}_i^0\cdot\overset{\circ}{\mathbf{H}}\cdot\mathbf{q}_j^0=\mathbf{q}_i\cdot\widehat{\mathbf{H}}\cdot\mathbf{q}_j,\quad i,j=\overline{1,3}, \tag{2.6}$$

which follows immediately from (1.16) with allowance for (1.4). The direction of the tangent to the path constructed from the components of $\overset{\circ}{\mathbf{H}}$ in the basis $\mathbf{q}_i^0$, is established from the components of the material derivative of $\overset{\circ}{\mathbf{H}}$ in the same basis.

THEOREM 2. The direction of the tangent to the strain path constructed from the components $\mathbf{q}_i\cdot\widehat{\mathbf{H}}\cdot\mathbf{q}_j$, coincides with the vector $\mathfrak{b}^Z$, found from the components of the Zaremby derivative $\widehat{\mathbf{h}}^Z$ in the basis $\mathbf{q}_i$. With allowance for Theorem 1 and the above observation, proof of Theorem 2 reduces to derivation of the relation

$$\mathbf{q}_i^0\cdot\dot{\overset{\circ}{\mathbf{H}}}\cdot\mathbf{q}_j^0=\mathbf{q}_i\cdot\widehat{\mathbf{H}}^Z\cdot\mathbf{q}_j,\quad i,j=\overline{1,3}. \tag{2.7}$$

In fact, using (1.3) and (1.13)-(1.15), we have

$$\mathbf{q}_i^0\cdot\dot{\overset{\circ}{\mathbf{H}}}\cdot\mathbf{q}_j^0=\mathbf{q}_i\cdot\mathbf{R}^T\cdot\dot{\overset{\circ}{\mathbf{H}}}\cdot\mathbf{R}\cdot\mathbf{q}_j=\mathbf{q}_i\cdot\left\{\sum_{k=1}^{3}\dot{\widehat{H}}_{k}\widehat{\mathbf{p}}^k\widehat{\mathbf{p}}_k-(\boldsymbol{\Omega}-\boldsymbol{\Omega}_V)\cdot\widehat{\mathbf{H}}+\widehat{\mathbf{H}}\cdot(\boldsymbol{\Omega}-\boldsymbol{\Omega}_V)\right\}\cdot\mathbf{q}_j=\mathbf{q}_i\cdot\widehat{\mathbf{H}}^Z\cdot\mathbf{q}_j.$$

Thus, the image of the loading process [13] determined from the components of $\boldsymbol{\sigma}$ and $\widehat{\mathbf{H}}$ in the basis $\mathbf{q}_i$ of the moving system $M\zeta^1\zeta^2\zeta^3$ is indifferent. In accordance with the definition of the tangent to the strain path, the elementary length of an arc of the latter

$$ds=(\mathfrak{b}^Z\cdot\mathfrak{b}^Z)^{1/2}dt=(\widehat{\mathbf{h}}^Z:\widehat{\mathbf{h}}^Z)^{1/2}dt. \tag{2.8}$$

Then the length of the arc $\forall t$

$$s=\int_{t_0}^{t}(\mathfrak{b}^Z\cdot\mathfrak{b}^Z)^{1/2}d\tau=\int_{t_0}^{t}(\widehat{\mathbf{h}}^Z:\widehat{\mathbf{h}}^Z)^{1/2}d\tau. \tag{2.9}$$

The angle of convergence [7] ϑ is found from the relation

$$\cos\vartheta=\frac{\boldsymbol{\Sigma}\cdot\mathfrak{b}^Z}{|\boldsymbol{\Sigma}|\,|\mathfrak{b}^Z|}=\frac{\mathrm{S}:\widehat{\mathbf{h}}^Z}{(\mathrm{S}:\mathrm{S})^{1/2}(\widehat{\mathbf{h}}^Z:\widehat{\mathbf{h}}^Z)^{1/2}}. \tag{2.10}$$

All of the parameters of the curvature and twist of the strain path are also naturally determined for the path constructed in terms of the movable coordinate system $M\zeta^1\zeta^2\zeta^3$.

3. Governing Relations. Of course, the proposed constitutive equations must satisfy the general requirements (postulates) established in the theory of governing relations [8, 15]. However, there are some examples of the construction of physical equations which satisfy all of the postulates but fail to offer a realistic description of elastoplastic strain processes (see [3], for example). Thus, here we have imposed limitations on the form of the constitutive equations subject to the above general postulates) and we introduce several additional requirements of a structural nature:

a) in the case of small displacement gradients, the physical equations of the NTP coincide with the relations of special theories of plasticity formulated on the basis of the theory of elastoplastic processes [6, 7, 13];

b) the measures used for the stress and strain states and the rates of their change permit the normal expansion (in the form of the sum of spherical and deviatoric components) into parts responsible for the change in volume and form;

c) the objective measures used for the stress and strain intensities are corotational, i.e. represent the relative rates of change in the stresses and strains determined by the same observer (moving or stationary);

d) in the construction of the image of the loading proceeds in terms of the actual configuration, the movable coordinate system $M\zeta^1\zeta^2\zeta^3$ is chosen so that the motion of the system $M\zeta^1\zeta^2\zeta^3$ characterizes the quasirigid (nondeformational) displacement of the same set of three material fibers fixed in $\mathscr{K}_0$. We note again that in the general case of motion of a deformable medium, this displacement is not actually realized and is regarded only as a method of decomposing the motion. Here, after movement of the undeformed set of three fibers from the position it occupies in $\mathscr{K}_0$, to the position it occupies in $\mathscr{K}_t$ (which coincides with $M\zeta^1\zeta^2\zeta^3$) and after deformation, the fibers are no longer orthogonal to each other and are generally oriented at random relative to $M\zeta^1\zeta^2\zeta^3$. The components of the chosen strain measure, fixed at an arbitrary moment of time t by an observe, in the system $M\zeta^1\zeta^2\zeta^3$. and characterizing the change in the lengths of and angles between the fibers of the three-fiber set (during the entire loading process), give the image point of the strain path in the five-dimensional deviatoric (or isomorphous vector) space. It should be noted that requirements "b" and "d" were used in Part 2 to introduce the image of the loading process.

Here, we are examining governing relations of the rate type. For different cases of deformation (over paths with a break or with small or moderate curvature), we can write the same type of relations [10, 16]:

$$\boldsymbol{\sigma}^Z = \mathbf{F}_1 : \hat{\mathbf{H}}^Z + \mathbf{B}\dot{\Theta} + \mathbf{R}, \tag{3.1}$$

where $\mathbf{F}_1$ is a tetravalent indifferent property tensor; θ is the temperature; $\mathbf{B}$ and $\mathbf{R}$ are divalent indifferent tensors. The specific expressions of the tensors $\mathbf{F}_1$, $\mathbf{B}$, and $\mathbf{R}$ are found in accordance with the type of strain path and are presented in [16]. For example, in the case of isothermal deformation over a path of small curvature, the tensor $\mathbf{B}$ is a zero tensor and

$$\mathbf{F}_1 = \frac{1}{2}\left[\Phi'(s) - \Phi(s)\dot{\hat{H}}_i^Z/(\hat{H}_i^Z)^2\right](\mathbf{C}_{II} + \mathbf{C}_{III}) + \frac{1}{3}\left[K_1 - \Phi'(s) + \Phi(s)\dot{\hat{H}}_i^Z/(\hat{H}_i^Z)^2\right]\mathbf{C}_I; \tag{3.2}$$

$$\mathbf{R} = \Phi(s)\hat{\mathbf{h}}^{ZZ}/\hat{H}_i^Z. \tag{3.3}$$

Here, $\Phi(s)$ is a universal function of the material; $\Phi'(s) = d\Phi/ds$; $\mathbf{C}_I, \mathbf{C}_{II}, \mathbf{C}_{III}$. are fourth-rank isotropic tensors [9]; the superscripts $\cdot$, Z, and ZZ denote the material derivative, the Zaremby derivative, and the double Zaremby derivative; $\hat{\mathbf{h}}^{ZZ} = \dot{\hat{\mathbf{h}}}^Z - \boldsymbol{\Omega}\cdot\hat{\mathbf{h}}^Z + \hat{\mathbf{h}}^Z\cdot\boldsymbol{\Omega}$; $K_1 = Ke^{I_1(\hat{H})}$ (K is the compressive bulk modulus); $\hat{H}_i^Z = (\hat{\mathbf{h}}^Z : \hat{\mathbf{h}}^Z)^{1/2}$; $\dot{\hat{H}}_i^Z = (\hat{\mathbf{h}}^Z : \dot{\hat{\mathbf{h}}}^Z)/(\hat{H}_i^Z)$.

It is not hard to see that the above governing relations satisfy the above-noted additional requirements. To determine the validity of proposed physical equations, it will generally be necessary to conduct experimental studies of complex loading with large strains. Unfortunately, such studies are not yet being done. We thus turn to a sample problem involving simple shear.

4. Results of Solution of Sample Problem. We will concern ourselves mainly with models of quasielastic materials. One of the first models of this type was the relation proposed by Truesdell, Prager, and Green [15, 17, 18] for a hyperelastic material

$$\boldsymbol{\sigma}^J = \boldsymbol{\Pi} : \mathbf{D}_{\varepsilon} \tag{4.1}$$

where J is a derivative in the Jaumann–Noll sense; $\boldsymbol{\Pi}$ is a fourth-rank property tensor with constant components. As was noted in several studies [3, 10, etc], the components of the stress tensor (in the basis of the system of reference) undergo oscillations in the solution of the problem of simple shear with the use of physical equation (4.1). Such behavior does not correspond to monotonic loading. On the basis of this, most investigators [3, 12, 19, etc.] have deemed use of the Jaumann derivative in the governing relations to be unacceptable. The following is proposed as an alternative relation:

$$\boldsymbol{\sigma}^Z = \boldsymbol{\Pi} : \mathbf{D}. \tag{4.2}$$

Results obtained for the shear problem using physical equation (4.2) are presented in Fig. 1 (here and below, the components of the stress tensor σ are determined in the orthonormalized Cartesian basis of the system of reference). It can be seen from the figure that the results of the solution are monotonic in character in this case. However, as the parameter t approaches infinity, the deformation process becomes equivalent to compression by smooth slabs in the direction of the x^2 axis and tension along the x^1 axis. It should be expected that in this case $\sigma_{11} \to \infty$, $\sigma_{22} \to -\infty$, $\sigma_{12} \to 0$, which does not correspond to the solution based on (4.2).

We note that relations of the form (4.1) and (4.2) do not meet the "c" requirement of corotation. Figure 2 gives analogous results obtained using the relations

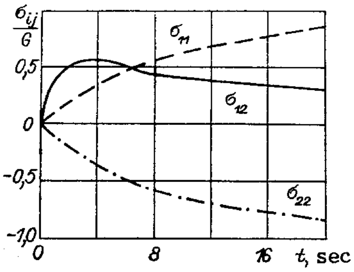

Fig. 3

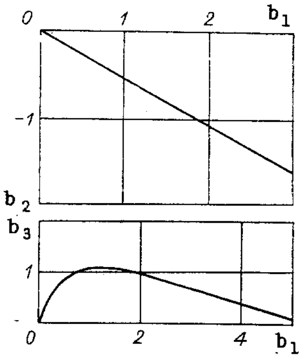

Fig. 4

$$\boldsymbol{\sigma}^J = \boldsymbol{\Pi} : \widehat{\mathbf{H}}^J,\ \boldsymbol{\sigma}^Z = \boldsymbol{\Pi} : \widehat{\mathbf{H}}^Z, \tag{4.3}$$

satisfying this requirement. Here, the solutions coincide identically, which is due to the isotropy of the tensor $\boldsymbol{\Pi}$. It is evident from Fig. 2 that the functions $\sigma_{11}(t)$, $\sigma_{22}(t)$ correspond qualitatively to those presented in Fig. 1, but the difference in the behavior of $\sigma_{12}(t)$ is quite significant.

Here, the relation $\sigma_{12}(t)$ found on the basis of (4.3) more closely reflects the actual deformation process in question.

Figure 3 shows similar results for the elastoplastic model (small-curvature theory, with the physical relations having the form (3.1-3.3)). Figure 2 shows the character of the change in the components of the stress tensor for a quasielastic material. The strain path shown in Fig. 4, constructed from the components of the Hencky tensor in the basis $\mathbf{q}_i$, confirms the validity of using small-curvature theory for the loading process being considered.

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DETERMINATION OF THE DEPTH OF THE PLASTIC REGION IN THE PRESSURE OF A FLAT DIE ON A HALF-PLANE

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Let a flat die of the length ℓ be pressed into a rectilinear boundary without friction so that a pressure distribution q is created under the die. Such a problem was first examined by Prandtl with the assumption that the stresses were continuous everywhere except at the ends of the die and with the use of a constant yield condition (see [1] for example).

Let σ_1 and σ_2 be the principal stresses in the plane (x, y). By means of $2p = \sigma_1 + \sigma_2$ and $2\tau = \sigma_1 - \sigma_2$, any yield condition for an isotropic material can be written in the form $\tau = \tau(p)$.

Using ψ to designate the angle between the first principal direction and the x axis, we express the components of the stress tensor through p, τ, and ψ:

$$\sigma_x = p + \tau\cos 2\psi,\ \sigma_y = p - \tau\cos 2\psi,\qquad \sigma_{xy} = \tau\sin 2\psi. \tag{1}$$

Having inserted the equilibrium equation into (1), we obtain the following system of equations:

$$\begin{gathered}\partial(p + \tau\cos 2\psi)/\partial x + \partial(\tau\sin 2\psi)/\partial y = 0,\\ \partial(\tau\sin 2\psi)/\partial x + \partial(p - \tau\cos 2\psi)/\partial y = 0.\end{gathered} \tag{2}$$

We assume that $|\tau'| < 1$. Then system (2) is hyperbolic. Two families of characteristic curves and relations along these curves can be written for the system:

$$\begin{gathered}(\cos 2\psi + \tau')dy = (\sin 2\psi + \sqrt{1-(\tau')^2})dx,\\ \psi + \int\limits_{p_0}^{p} \frac{\sqrt{1-(\tau')^2}}{2\tau}\,d\xi = r - \text{const},\\ (\cos 2\psi + \tau')dy = (\sin 2\psi - \sqrt{1-(\tau')^2})dx,\\ \psi - \int\limits_{p_0}^{p} \frac{\sqrt{1-(\tau')^2}}{2\tau}\,d\xi = s - \text{const}.\end{gathered}$$

We will examine the stress field in the plane of the flows (Fig. 1) and the plane of the characteristics. The simplest stress field develops in region ABA_{11}: $p_0 = \tau(p_0) - q$, $\psi = 0$. This region corresponds to the origin of the coordinates in the characteristic plane. The region $A_{11}BA_{21}$ contains a simple centered s-wave, while region $AA_{11}A_{12}$ also contains a simple

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